Affinization of S^1

k = fixed ordinary ring. $(\mathbb{Z}_{(p)})$.

Recall: • $\mathrm{Stacks}_{/k} = \mathrm{Shv}_{\mathrm{Spaces}}(k\text{-}\mathrm{alg}^{op}_{fpqc})$.

• $f_*: \mathrm{Stacks}_{/k} \rightleftarrows \mathrm{Spaces} : f^*$, $f^* \dashv f_*$

$$F \longmapsto f_*F = F(k)$$

$$(f^*S)(R) = S \longleftarrow\!\shortmid S$$

• for any stack G, $R\Gamma(G,\mathcal{O}) := R\mathrm{Hom}(\mathbb{Z}[G], \mathbb{G}_a)$

$$= \underset{G(R)}{R\lim}\, R.$$

$$\rightsquigarrow \quad G \longrightarrow \underbrace{\mathrm{Spec}(R\Gamma(G,\mathcal{O}))}_{}$$

universal affine stack receiving a map from G.

- X space, $f^*X \rightsquigarrow \mathrm{Aff}(X) = \mathrm{Spec}(R\Gamma(f^*X, \mathcal{O}))$.

$$R\mathrm{Hom}(\mathbb{Z}[f^*X], \mathbb{G}_a) = R\mathrm{Hom}_{\mathrm{Spaces}}(\mathbb{Z}[X], k)$$

$$= R\Gamma_{\mathrm{sing}}(X; k)$$

Ex $X = pt$. $\mathrm{Aff}(X) = \mathrm{Spec}(k)$

Check: • $f^* \mathbb{Z} = \underline{\mathbb{Z}}$ constant gp scheme.

• f^* comm. w/ colim, hence comm. w/ classifying stack construction.

$$f^*(S^1) = f^*(B\mathbb{Z}) = B(f^*\mathbb{Z}) = B(\underline{\mathbb{Z}}).$$

Recall: $G: W \times \mathbb{A}^1 \longrightarrow W \times \mathbb{A}^1 \quad / \mathbb{Z}_{(p)}$.

$$(x, a) \longmapsto (\mathrm{Frob}(x) - [a^{p-1}] \cdot x,\ a)$$

if $a = 1$, then $\underset{[1,0,0,\cdots]}{x=1} \in \ker(G) \underset{\mathbb{A}^1}{\times} \{1\}$.

$$\mathbb{Z} \longrightarrow \mathrm{Fix} = \ker G \underset{\mathbb{A}^1}{\times} \{1\} \qquad /\mathbb{Z}_{(p)} \ni k.$$

$$f^*(S^1) \simeq B(\mathbb{Z}) \longrightarrow B(\mathrm{Fix})$$

Recall: Last BH is relatively affine $/ [\mathbb{A}^1/\mathbb{G}_m]_{\mathbb{Z}_{(p)}}$.

$$\Longrightarrow (S^1_{\mathrm{Fil}})^u = (BH)^u = B\mathrm{Fix} \text{ is affine } /\mathbb{Z}_{(p)}$$

Prop: $k = \mathbb{Z}_{(p)}$, the map $f^* S^1 \longrightarrow B\mathrm{Fix} = (S^1_{\mathrm{Fil}})^u$ is the affinization map for $S^1/\mathbb{Z}_{(p)}$,

$$\Rightarrow \quad R\Gamma(B\mathrm{Fix}, \mathcal{O}) \xrightarrow{\simeq} R\Gamma_{\mathrm{sing}}(S^1; \mathbb{Z}_{(p)}).$$

Lemma: $R\Gamma(B\mathrm{Fix}, \mathcal{O}) \overset{L}{\underset{\mathbb{Z}_{(p)}}{\otimes}} A \overset{\checkmark}{\simeq} R\Gamma(B(\mathrm{Fix}_A), \mathcal{O})$

$$R\Gamma_{\mathrm{sing}}(S^1; \mathbb{Z}_{(p)}) \overset{L}{\underset{\mathbb{Z}_{(p)}}{\otimes}} A \overset{\checkmark}{\simeq} R\Gamma_{\mathrm{sing}}(S^1; A).$$

pf: $C^\cdot$ complex of left bounded, termwise flat $/R$

$$C^\cdot \overset{L}{\underset{R}{\otimes}} M = \text{total}\left(\text{termwise} \overset{L}{\underset{R}{\otimes}} M\right)$$

$(*)$

if M has finite Tor dim $/R$.

(non-)Ex: $k = \text{ring}$ $\quad R = k[\varepsilon]$.

$$k[0] = C = \left(k[\varepsilon] \xrightarrow{\cdot\varepsilon} k[\varepsilon] \xrightarrow{\cdot\varepsilon} k[\varepsilon] \longrightarrow \cdots\right)$$

and $C \overset{L}{\underset{R}{\otimes}} k \neq \left(\text{termwise} \underset{R}{\otimes} k\right)$

Lemma: the induced map

$$
\begin{tikzcd}
R\Gamma_{sing}(S^1; \mathbb{Q}) \arrow[r, equal] & \mathbb{Q} \oplus \mathbb{Q}[-1] \\
R\Gamma(B(\mathrm{Fix}_{\mathbb{Q}}); \mathcal{O}) \arrow[u, "\cong"] & \\
\simeq L\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-1]) & \\
\simeq \Lambda^*_{\mathbb{Q}}(\mathbb{Q}[-1]) & \\
= \mathbb{Q} \oplus \mathbb{Q}[-1] \arrow[uuuur, bend right, "id"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
W_{\mathbb{Q}} \arrow[r, equal] & \prod_{\mathbb{N}} \mathbb{G}_a & (1,1,1,\dots) \\
\mathrm{Fix}_{\mathbb{Q}} \arrow[u] \arrow[r, equal] & \mathbb{G}_a \arrow[u, "\Delta"'] & 1 \arrow[u, mapsto] \\
& \mathbb{Z} \arrow[u] & 1 \arrow[u, mapsto]
\end{tikzcd}
$$

Lemma: $R\Gamma(B\mathrm{Fix}_{\mathbb{F}_p}, \mathcal{O}) \xrightarrow{\simeq} R\Gamma_{\mathrm{syn}}(S^1; \mathbb{F}_p)$

in char p.

$$0 \longrightarrow \mathbb{Z}/p^n \longrightarrow W_n \xrightarrow{\mathrm{Frob}-\mathrm{id}} W_n \longrightarrow 0.$$

$$\mathbb{Z} \to \mathrm{Fix}_{\mathbb{F}_p}$$

$$\rightsquigarrow \mathrm{Fix}_{\mathbb{F}_p} = \lim_n (\mathbb{Z}/p^n)_{\mathbb{F}_p} = \mathrm{Spec}(\mathrm{Fun}_{\mathrm{cont}}(\mathbb{Z}_p, \mathbb{F}_p)).$$

so $R\Gamma(B\mathrm{Fix}_{\mathbb{F}_p}, \mathcal{O}) = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p, \mathbb{F}_p)$.

and the induced $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p, \mathbb{F}_p) \xrightarrow{\simeq} R\Gamma(\mathbb{Z}, \mathbb{F}_p)$

comes from $\mathbb{Z} \longrightarrow \mathbb{Z}_p$

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p, \mathbb{F}_p) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_p)$$

Lemma: if $M \in D(\mathbb{Z}_{(p)})$ satisfies

(1) $M \otimes^L_{\mathbb{Z}_{(p)}} \mathbb{F}_p = 0$;

(2) $M \otimes^L_{\mathbb{Z}_{(p)}} \mathbb{Q} = 0$.

Then $M = 0$.

pf:

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{\cdot p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

$$M \xrightarrow{\cdot p} M \longrightarrow \underbrace{M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p}_{=0} \xrightarrow{+1}.$$

$\Rightarrow$ $M \xrightarrow{\cdot p} M$ is an isom.

$$\operatorname{colim}\left(M \xrightarrow{\cdot p} M \xrightarrow{\cdot p} \cdots\right) = M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = 0.$$

$$\mathbb{Q} = \operatorname{colim}\left(\mathbb{Z}_{(p)} \xrightarrow{\cdot p} \mathbb{Z}_{(p)} \xrightarrow{\cdot p} \cdots\right)$$